Zusammenfassung

Dies hier soll eine Mitschrift der Vorlesung Mathematische Logik II im SS2003 bei Jacques Duparc und Achim Blumensath werden. Sie besteht bis auf weiteres nur aus meinen Mitschriften in der Vorlesung, daher gibt es ausdrücklich *keine* Garantie für Korrektheit und Vollständigkeit. Trotzdem ist es vielleicht nicht schlecht, falls jemand mal eine Vorlesung verpasst oder so. Selbstverständlich kann das Skript frei kopiert und weitergegeben etc. werden. Inhaltliche (!) Änderungsvorschläge und Korrekturen oder sonstiges Feedback? Immer her damit: Bastian.Schwittay@post.rwth-aachen.de.

Aachen, 22. Mai 2003

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0.1 Literaturtips

Cori, René ; Lascar, Daniel : Mathematical Logic
Kunun, Kenneth : Set Theory. An introduction to independence proofs

Kapitel 1

Set Theory

1.1 Introduction

Vorlesung am 23.04.03

- A theory of first-order logic with equality and language $\{\in\}$ where $\in$ is interpreted as "membership".
- Why formal logic?
  1. We need a precise notion of what is provable or not.
  2. Needed to precisely define properties.

A formula $\varphi(\bar{x})$, $\bar{x} = x_1, \ldots, x_n$, should be regarded as a precise unambiguos way of stating a property of $x_1, \ldots, x_n$.
Compare this with :

> "Let n be the least positive integer not definable by an English expression using forty words or less."

But we have just used less than forty words to define n.

The basic symbols of our formal language are:

$$\wedge, \neg, \exists, (,), \in, =, \underbrace{v_i \quad (i \in \mathbf{N})}_{variables}$$

A formula is any expression constructed by the rules:

1. $v_i \in v_j, v_i = v_j \quad (\forall i, j \in \mathbf{N})$
2. If Φ and Ψ are formulas, so are $(\Phi) \wedge (\Psi)$, $\neg(\Phi)$, $\exists v_i(\Phi) \quad (i \in \mathbf{N})$

We use abbreviations:

1. $\forall v_i(\Phi)$ abbreviates $\neg(\exists v_i(\neg(\Phi)))$
2. $(\Phi) \vee (\Psi)$ abbreviates $\neg((\neg(\Phi) \wedge (\neg(\Psi)))$
3. $(\Phi) \to (\Psi)$ abbreviates $(\neg(\Phi)) \vee (\Psi)$
4. $(\Phi) \leftrightarrow (\Psi)$ abbreviates $((\Phi) \to (\Psi)) \wedge ((\Psi) \to (\Phi))$
5. Parentheses are dropped if it is clear from the context how to put them back.

6. $v_i \notin v_j$ abbreviates $\neg(v_i \in v_j)$
   $v_i \neq v_j$ abbreviates $\neg(v_i = v_j)$

7. Other letters and subscripted letters from the English, Greek and Hebrew alphabets are used for variables.

A *subformula* of Φ is a consecutive sequence of symbols of Φ which forms a formula. Example:

$$\underbrace{(\overbrace{\exists v_0 \underbrace{(v_0 \in v_1)}} \wedge \overbrace{(\exists v_2 \underbrace{(v_2 \in v_1)})})} \quad \text{5 subformulas}$$

The *scope* of an occurence of a quantifier $\exists v_i$ is the unique subformula beginning with that $\exists v_i$:

$$(\exists v_0 \quad \underbrace{(v_0 \in v_1)}_{\text{scope of } \exists v_0} \quad \wedge(\exists v_2 \quad \underbrace{(v_2 \in v_1)}_{\text{scope of } \exists v_2} \quad)$$

An occurence of a variable in a formula is called *bound* if it lies in the scope of a quantifier acting on that variable. Otherwise it is called *free.*

Intuitively a formula $\Phi(x_1, \ldots, x_n)$ expresses a property of its free variables $x_1, \ldots, x_n$.

A *sentence* is a formula with no free variables; intuitively it is either true or false. An *axiom* is a sentence.

Set Theory: ZFC is a certain set (collection) of axioms.

- If S is a set of sentences and Φ is a sentence

  $S \vdash \Phi$ stands for: Φ is provable from S.

- If S is a set of sentences and Φ is a formula

  $S \vdash \Phi$ means $S \vdash \Phi'$ where Φ' is the *universal closure* of Φ, i.e. if the free variables of Φ are $x_1, \ldots, x_n$ then $\Phi' = \forall x_1 \ldots \forall x_n \Phi$.

- If S is a set of sentences, S is *consistent* if for no Φ, $S \vdash \Phi$ *and* $S \vdash \neg(\Phi)$ is true.

- If S is *inconsistent* then $S \vdash \Psi$ for any formula Ψ, hence it is totally useless.

Theorem:

1. If $S \vdash \Phi$, then there exists a finite $S_0 \subseteq S$ such that $S_0 \vdash \Phi$.

2. If S is inconsistent, there is a finite $S_0 \subseteq S$ such that S_0 is inconsistent.

1.2 ZFC - Axioms

Vorlesung am 29.04.03

AX 0: Set Existence $\exists x(x = x)$

AX 1: Extensionality $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$
This says a set is determined by its members: $\{0, 1, 0, 0\} = \{0, 1\} = \{1, 0\}$

AX 3: Comprehension Scheme
The first idea would be to say

$$\exists y \forall x(x \in y \leftrightarrow \Phi)$$

where Φ is a formula where y is not free. Typically $\Phi = \Phi(x, w_1, \ldots, w_n)$.
But this way one gets the Russell Paradox. Take $\Phi : x \notin x$.

$$\exists y \forall x(x \in y \leftrightarrow x \notin x)$$

So for this particular y we get $y \in y \leftrightarrow y \notin y$.

For each formula Φ without y free, the universal closure of the following is an axiom:

$$\exists y \forall x(x \in y \leftrightarrow (x \in z \wedge \Phi))$$

Such a set is unique by extensionality. For commodity we write this set
$y = \{x \in z : \Phi\}$

By Axiom 0 take a set z.
By *Comprehension* $y = \{x \in z : x \neq x\}$ exists , it is unique by extensionality and we denote it $\emptyset$. In fact, with Axioms 0,1 and 3 this theory cannot refute $\forall x(x = \emptyset)$.

We can prove that there exists no universal set:
Theorem: $\neg \exists z \forall x(x \in z)$
Proof: Towards a contradiction, we assume that there exists such z. By Comprehension $y = \{x \in z : x \notin x\} = \{x : x \notin x\}$. This is a contradiction (Russell Paradox).$\square$

AX 4: Pairing

$$\forall x \forall y \exists z(x \in z \wedge y \in z)$$

By comprehension we can form $A = \{a \in z : a = x \vee a = y\}$. It is the only set (by extensionality) that contains x and y. We denote it $\{x, y\}$.
$\langle x, y\rangle$ is defined by

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

This set exists by Pairing. To see that $\langle x, y\rangle$ is an ordered pair we need to check $\forall x \forall y \forall x' \forall y'(\langle x, y\rangle = \langle x', y'\rangle \rightarrow x = x' \wedge y = y')$.

AX 5: Union

$$\forall \mathcal{F} \exists A \forall Y \forall x((x \in Y \wedge Y \in \mathcal{F}) \rightarrow x \in A)$$

Notation: $C \subseteq B$ abbreviates $\forall x(x \in C \rightarrow x \in B)$. This justifies

$$\bigcup \mathcal{F} = \{x : \exists Y \in \mathcal{F}(x \in Y)\}.$$

This set is unique by extensionality.
Ex.:

$$\bigcup \emptyset = \emptyset$$

$$\bigcup\{\emptyset\} = \emptyset$$
$$\bigcup\{\{\emptyset\}\} = \{\emptyset\}$$
$$\bigcup\{\emptyset, \{\emptyset\}\} = \{\emptyset\}$$

We write $A \cup B$ for $\bigcup\{A, B\}$.

When $\mathcal{F} = \emptyset$
$$\bigcap \mathcal{F} = \{x : \forall Y \in \mathcal{F}(x \in Y)\}$$

Finally we write:
$$A \cap B = \bigcap\{A, B\}$$

$$A - B = \{x \in A : x \notin B\}.$$

AX 6: Replacement Scheme

For each formula Φ without y free the following is an axiom:

$$\forall x \in A \exists! y \Phi(x, y) \rightarrow \exists Y \forall x \in A \exists y \in Y \Phi(x, y).$$

By Replacement and Comprehension $\{y : \exists x \in A \Phi(x, y)\}$ exists, since it is also $\{y \in Y : \exists x \in A \Phi(x, y)\}$ for any Y such that $\forall x \in A \exists y \in Y \Phi(x, y)$.

For any A and B we can form

$$A \times B = \{\langle x, y\rangle : x \in A \wedge y \in B\} \qquad \text{Exercise.}$$

A *binary* relation is a set R all of whose elements are ordered pairs.

$dom(R) = \{x : \exists y(\langle x, y\rangle \in R)\}$

$ran(R) = \{y : \exists x(\langle x, y\rangle \in R)\}$

Exercise: show that if R exists, then $dom(R)$ and $ran(R)$ also exist.

We define: $R^{-1} = \{\langle x, \rangle : \langle y, x\rangle \in R\} \qquad (R^{-1})^{-1} = R.$

f is a function, if f is a relation *and* $\forall x \in dom(f) \exists! y \in ran(f)(\langle x, y\rangle \in f)$.

$f : A \longrightarrow B$ means: f is a function, $A = dom(f), ran(f) \subseteq B$.

- If $f : A \longrightarrow B$ $f(x)$ is the unique y such that $\langle x, y\rangle \in f$.
- If $C \subseteq A$, $f \upharpoonright C = f \cap C \times B$.
- If $C \subseteq A$, $f''C = ran(f \upharpoonright C) = \{f(x) : x \in C\}$.
- $f : A \longrightarrow B$ is *1-1* (or injective) iff f^{-1} is a function.
- $f : A \longrightarrow B$ is *onto* (or surjective) if $ran(f) = B$.
- $f : A \longrightarrow B$ is bijective, if it is 1-1 and onto.

A *total ordering* is a pair $\langle A, R\rangle$ such that R totally orders A, i.e.

- A is a set
- R is a relation
  - transitive: $\forall x \forall y \forall z \in A(xRy \wedge yRz \rightarrow xRz)$
  - trichotomy holds: $\forall x \forall y \in A(x = y \vee xRy \vee yRx)$
  - irreflexive: $\forall x \in A \neg(xRx)$.

As usual we write xRy for $\langle x, y\rangle \in R$.

We say $\langle A, R\rangle$ is a *well-ordering* (or R well-orders A), if $\langle A, R\rangle$ is a total ordering and every non-empty subset of A has an R-least element.

Ex.: $<$ well-orders $\mathbf{N}$.
If $\langle A, R\rangle$ is a well-ordering, so is $\langle B, R\rangle$ for any $B \subseteq A$.

Definition: Let A, B be sets and R, S be relations.

$\langle A, R\rangle \cong \langle B, S\rangle$ iff there exists a bijection $f : A \longrightarrow B$ such that $\forall x, y \in A(xRy \leftrightarrow f(x)Sf(y)$. f is called isomorphism.

Remark: If $f : A \longrightarrow B$ is an isomorphism, f^{-1} is also an isomorphism.

Notation: Let $\langle A, <_R\rangle$ be a well-ordering. If $x \in A : pred_{<_R}(x, A) = \{y \in A : y <_R x\}$.

Basic Properties of well-orderings:
Lemma(1): If $\langle A, <_R\rangle$ is a well-ordering, then for all $x \in A$:

$$\langle A, <_R\rangle \not\cong \langle pred_{<_R}(x, A), <_R\rangle$$

Proof: If $f : A \longrightarrow pred_{<_R}(x, A)$ was an isomorphism, consider the $<_R$-least element of $\{y \in A : f(y) \neq y\}$ (this set is not empty since $f(x) \neq x$).
Take a, the $<_R$-least. We discuss $f(a)$:

1. $a = f(a)$: impossible

2. $f(a) <_R a$ implies $f(f(a)) <_R f(a)$, since f is an isomorphism. Then $f(a) <_R f(a)$, which is a contradiction.

3. $a <_R f(a)$: then use f^{-1}. $f^{-1} <_R f^{-1}(f(a)) = a$. Because $f^{-1} <_R a, f(f^{-1}(a)) = f^{-1}(a)$, so $a = f^{-1}(a)$, which implies $f(a) = a$, contradiction. $\square$

Vorlesung am 30.04.03

Lemma(1): If $\langle A, <_R \rangle$ and $\langle B, <_S \rangle$ are isomorphic then the isomorphism is *unique*.

Proof: If A and B are empty, there exists only one function $f : \emptyset \longrightarrow \emptyset$.
If A is not empty, towards a contradiction we assume f, g are two different isomorphisms: take $a \in A$ the $<_R$-least element s.t. $f(a) = g(a)$.

There are two possibilities:

1. $f(a) <_S g(a)$
   Take $b =$ the unique element s.t. $g(b) = f(a)$ (i.e. $b = g^{-1} \circ f(a)$).
   $f(a) <_S g(a)$ implies $g(b) <_S g(a)$ implies $b <_R a$ implies $f(b) <_S f(a)$ implies $f(b) <_S g(b)$. In particular $f(a) \neq g(b)$. This contradicts the $<_R$-minimality of a.
2. $g(a) <_S f(a)$
   Same proof switching f and g.

Theorem 1: Let $\langle A, <_R \rangle$, $\langle B, <_S \rangle$ be two well-orderings then exactly one of the following holds:

1. $\langle A, <_R \rangle \cong \langle B, <_S \rangle$
2. $\exists y \in B \langle A, <_R \rangle \cong \langle pred_{<_S}(y, B), <_S \rangle$
3. $\exists y \in A \langle pred_{<_R}(y, A), <_R \rangle \cong \langle B, <_S \rangle$

Proof: Set
$f = \{\langle v, w \rangle : v \in A \wedge w \in B \wedge \langle pred_{<_R}(v, A), <_R \rangle \cong \langle pred_{<_S}(w, B), <_S \rangle\}$.

- **a)** f is a bijection from $dom(f)$ to $ran(f)$.
  1. f is obviously onto $ran(f)$.
  2. f is 1-1.
     TAC we assume there exists $v, v' \in dom(f)$ s.t. $f(v) = f(v')$ and $v \neq v'$. By symmetry let us assume $v <_R v'$. Then both these hold:

     $$\langle pred_{<_R}(v, A), <_R \rangle \cong \langle pred_{<_S}(f(v), B), <_S \rangle$$

     $$\langle pred_{<_R}(v', A), <_R \rangle \cong \langle pred_{<_S}(f(v'), B), <_S \rangle.$$

     So by composition of isomorphism:

     $$\langle pred_{<_R}(v, A), <_R \rangle \cong \langle pred_{<_R}(v', A), <_R \rangle.$$

     But $pred_{<_R}(v, A) = \{y \in A : y <_R v\} = \{y \in pred_{<_R}(v', A) : y <_R v\} = \langle pred_{<_R}(v, pred_{<_R}(v', A)), <_R \rangle$.
     Set $C = pred_{<_R}(v', A)$: $\langle pred_{<_R}(v, C), <_R \rangle \cong \langle C, <_R \rangle$ and $v \in C$, a contradiction to Lemma 1.
- **b)** f is an isomorphism: we must show for all $v, v' \in dom(f)$: $v <_R v' \rightarrow f(v) <_S f(v')$.
  Take $v <_R v'$, set $f(v) = w, f(v') = w'$.
  1. $w = w'$ is impossible, since f is 1-1.
  2. If we assume $w' <_S w$ then take the unique isomorphism φ between $\langle pred_{<_R}(v', A), <_R \rangle$ to $\langle pred_{<_S}(w', B), <_S \rangle$.

$\varphi \restriction pred_{<_R}(v, A)$ is an isomorphism between

$$\langle pred_{<_R}(v, A), <_R\rangle \quad \text{and} \quad \langle pred_{<_S}(\varphi(v), B), <_S\rangle.$$

$\varphi(v) <_S w' <_S w$. So $\varphi(v) <_S w$ hence

$$\langle pred_{<_S}(\varphi(v), B), <_S\rangle \cong \langle pred_{<_S}(w, B), <_S\rangle.$$

Contradiction to Lemma 1.

We have shown that $f : dom(f) \longrightarrow ran(f)$ is an isomorphism. It still remains to show that $dom(f)$ and $ran(f)$ are initial segments, i.e. if $v' \in dom(f)$ and $v <_R v'$ then $v \in dom(f)$.
Obviously since given the unique isomorphism φ between

$$\langle pred_{<_R}(v', A), <_R\rangle \quad \text{and} \quad \langle pred_{<_S}(\varphi(v'), B), <_S\rangle,$$

$\varphi \restriction pred_{<_S}(v, A)$ is an isomorphism between

$$\langle pred_{<_R}(v, A), <_R\rangle \quad \text{and} \quad \langle pred_{<_S}(\varphi(v), B), <_S\rangle,$$

so $\langle v, \varphi(v)\rangle \in A$, so $v \in dom(f)$; to show that $ran(f)$ is an initial segment - same proof.

Finally we show that $dom(f)$ and $ran(f)$ cannot be both proper subsets of respectively A and B.
TAC we assume $dom(f) \subset A$ and $ran(f) \subset B$ (proper subsets):
There exists $v \in A$: the $<_R$-least element not in $dom(f)$.
There exists $w \in B$: the $<_S$-least element not in $ran(f)$.
Then $dom(f) = pred_{<_R}(v, A)$ and $ran(f) = pred_{<_S}(w, B)$.
But f is an isomorphism between

$$\langle pred_{<_R}(v, A), <_R\rangle \quad \text{and} \quad \langle pred_{<_S}(w, B), <_S\rangle,$$

so by very definition of f, $\langle v, w\rangle \in A$. So $v \in dom(f)$ and $w \in ran(f)$, a contradiction.□

AX. 9: Axiom of Choice

$$\forall A \exists R(Rwell - ordersA)$$

1.3 Ordinals

Definition: A set x is transitive iff every element of x is a subset of x:
$\forall x \forall y(z \in y \in x \to z \in x)$ or $\forall a \in x(a \subseteq x)$.

Ex.: $\emptyset, \{\emptyset, \{\emptyset\}\}$ are transitive, $\{\{\emptyset\}\}$ is not transitive. **Definition:** x is an ordinal iff

- x is transitive
- x is well-ordered by $\in$

Formally "well-ordered by $\in$ " means $\in_x = \{\langle y, z\rangle \in x \times x : y \in z\}$ and $\in_x$ well-orders x.

Ex.: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ are ordinals.

$\{\{\{\emptyset\}\}, \{\emptyset\}, \emptyset\}$ is not an ordinal because $\in$ does not well-order this set: $\emptyset \notin \{\{\emptyset\}\}$ and $\emptyset \not\ni \{\{\emptyset\}\}$.

For simplicity we write given x, y ordinals:

- $x \cong y$ instead of $\langle x, \in_x \rangle \cong \langle y, \in_y \rangle$.
- $pred(x, y)$ instead of $pred_{\in_x}(x, y)$.

Theorem 2:

1. If x is an ordinal and $y \in x$ then
   (a) y is an ordinal
   (b) $y = pred(y, x)$.
2. If x and y are ordinals and $x \cong y$ then $x = y$.
3. If x and y are ordinals, one of the following holds:
   (a) $x = y$
   (b) $x \in y$
   (c) $y \in x$
4. If x, y, z are ordinals and $x \in y$ and $y \in z$ then $x \in z$.
5. If C is a non-empty set of ordinals, then $\exists x \in C \forall y \in C (x \in y \vee x = y)$.

Proof:

1. (a) i. We must show y is transitive, i.e. for any $a, b : a \in b \in y \to a \in x$.
   First $b \in y \in x \to b \in x$ and $a \in b \in x \to a \in x$.
   So a, y both belong to x. Now $\in$ well-orders x. So either $a \in y$ or $y \in a$.
   If $y \in a$ then $y \in a \in b \in y \to y \in y$, which contradicts the irreflexibility of $\in$. So it remains $a \in y$.
   ii. y is well-ordered by $\in$. Since x is an ordinal and $y \in x, y \subseteq x$. Since $\in$ well-orders x it well-orders any subset of x, in particular y.
   (b) is just definition:
   $y \subseteq pred(x, y)$: if $a \in y$ then $a \in pred(x, y)$, since $a \in y \in x \to a \in x$.
   $(\supseteq)$ if $a \in pred(y, x)$ then $a \in y$.
2. Let φ be the unique isomorphism between $\langle x, \in_x \rangle$ and $\langle y, \in_y \rangle$.
   If φ is the identity we are done by Ext.
   If φ is not the identity let a be the $\in$-least element in x s.t. $\varphi(a) \neq a$:
   (a) $b \in a \to b = \varphi(b) \in \varphi(a)$ hence $a \subseteq \varphi(a)$.
   (b) $c \in \varphi(a) \to \varphi^{-1}(c) \in a$ hence $a \supseteq \varphi(a)$.

   A Contradiction by Ext.
3. Use 1., 2., and Theorem 1 to show that at least one of the three conditions holds. That no more than one holds follows from the fact that no ordinal number can be a member of itself, since $x \in x$ would imply that $\in_x$ is not a (strict) total ordering (since $x \in_x x$).
4. is immediate by definition.
5. Note that the conclusion is , by 3., equivalent to $\exists x \in C (x \cap C = \emptyset)$. Let $x \in C$ be arbitrary. If $x \cap C \neq \emptyset$, then, since x is well-ordered by $\in$, there is an $\in$-least element x' of $x \cap C$, then $x' \cap C = \emptyset$.$\square$

Vorlesung am 06.05.03

- $\langle A, <_R \rangle < \langle B, <_S \rangle$ gdw. $\langle A, <_R \rangle \cong \langle pred_{<_S}(x, B), S \rangle$.
- $x < y$ gdw. $x \in y$.
- Ordinalzahlen werden durch $\in$ wohlgeordnet.

Theorem 3: Die Menge aller Ordinalzahlen existiert nicht:

$$\neg \exists z \forall x (x \quad \text{Ordinalzahl} \rightarrow x \in z)$$

Beweis: Sonst gäbe es die Menge

$$\mathbf{ON} = \{x : x \quad \text{Ordinalzahl}\}$$

- **ON** ist transitiv
  $x \in y \in \mathbf{ON} \Rightarrow x \in \mathbf{ON}$ (Theorem 2, 1.)
- **ON** wird durch $\in$ wohlgeordnet.
- $\Rightarrow$ **ON** ist eine Ordinalzahl.
- $\Rightarrow$ $\mathbf{ON} \in \mathbf{ON}$, Widerspruch: $\in$ ist irreflexiv fr Ordinalzahlen. □

Lemma: Ist α eine Menge von Ordinalzahlen mit $x \in y \in \alpha \Rightarrow x \in \alpha$, dann ist α eine Ordinalzahl.

Beweis:

- α ist nach Voraussetzung transitiv
- α wird durch $\in$ wohlgeordnet:
  - je zwei Elemente von α sind vergleichbar bzgl. $\in$
  - jede nichtleere Teilmenge von α besitzt ein $\in$-minimales Element.□

Theorem 4: Ist $\langle A, <_R \rangle$ eine Wohlordnung, dann gibt es eine eindeutige Ordinalzahl α mit $\langle A, <_R \rangle \cong \alpha$.
α heißt der *Typ* von $\langle A, <_R \rangle$.

Beweis:

- Eindeutigkeit folgt aus Theorem 2, 2.
- Existenz: Fr $a \in A$ sei $f(a)$ eine Ordinalzahl mit $pred_{<_R}(a, A) \cong f(a)$, falls so eine Zahl existiert.
  Sei $B := dom(f)$ und $\alpha := ran(f)$.

Beh.: α ist eine Ordinalzahl.
Nach dem Lemma reicht es zu zeigen, dass $x \in y \in \alpha \Rightarrow x \in \alpha$ gilt.
Sei $y \in \alpha$. Dann gibt es ein $a \in A$ mit $pred_{<_R}(a, A) \cong y$.
Sei φ der entsprechende Isomorphismus.
Setze $b := \varphi^{-1}(x) \Rightarrow b \in B$ und $pred_{<_R}(b, a) \cong x \Rightarrow (b, x) \in f \Rightarrow x \in ran(f)$.

Beh.: $f : \langle B, <_R \rangle \cong \alpha$. Beweis genau wie in Theorem 1.
Es bleibt zu zeigen: $A = B$.
Wenn nicht, dann existiert $a \in A$ mit $\alpha \cong \langle B, <_R \rangle \cong \langle pred_{<_R}(a, A), <_R \rangle \Rightarrow a \in B$ und $f(a) = \alpha \Rightarrow \alpha \in ran(f) = \alpha$. Widerspruch. □

Notation:

- Kleine griechische Buchstaben $\alpha, \beta, \gamma, \delta$ stehen immer für Ordinalzahlen, d.h. wir schreiben
$$\forall\alpha\Phi \quad \text{statt} \quad \forall\alpha(\alpha \quad \text{Ordinalzahl} \rightarrow \Phi$$
- $\alpha < \beta$ für: $\alpha \in \beta$
- $\alpha \leq \beta$ für: $\alpha \in \beta$ oder $\alpha = \beta$.

Definition: Sei X eine Menge von Ordinalzahlen:

- $sup(X) := \bigcup X$
- $min(X) := \bigcap X$, falls $X \neq \emptyset$.

Lemma:

1. $\forall\alpha\forall\beta(\alpha \leq \beta \leftrightarrow \alpha \subseteq \beta)$
2. Sei X eine Menge von Ordinalzahlen:
   - $sup(X)$ ist die kleinste Ordinalzahl $\geq$ allen Elementen von X.
   - $min(X)$ ist die kleinste Ordinalzahl in X (falls $X \neq \emptyset$).

Beweis:

1. trivial
   - $sup(X) = \{x : \exists y \in X(x \in y)\}$
   - $sup(X)$ ist eine Menge von Ordinalzahlen. Um zu zeigen, dass $sup(X)$ eine Ordinalzahl ist, reicht es zu zeigen, dass
     $a \in b \in sup(X) \rightarrow a \in sup(X)$.
     Sei $a \in b \in y \in X \Rightarrow a \in y \Rightarrow a \in sup(X)$.
     - **Beh.:** $sup(X)$ ist die kleinste Ordinalzahl $\geq$ allen Elementen aus X.
       * $a \in X \Rightarrow \alpha \subseteq sup(X) \Leftrightarrow \alpha \leq sup(X)$.
       * Sei β eine Ordinalzahl mit $\alpha \leq \beta$ für alle $\alpha \in X$.
         $\Rightarrow \forall\alpha \in X\alpha \subseteq \beta$
         $\Rightarrow sup(X) \subseteq \beta \Rightarrow sup(X) \leq \beta$.
   - Für $min(X)$: ähnlich. □

Definition: $S(\alpha) := \alpha \cup \{\alpha\}$. **Beispiel:**

$$\emptyset = \{\} \qquad (= 0)$$
$$S(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\} \qquad (= 1)$$
$$S(\{\emptyset\}) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} \qquad (= \{0, 1\} = 2)$$

Lemma: Sei α eine Ordinalzahl:

1. $S(\alpha)$ ist eine Ordinalzahl
2. $\alpha < S(\alpha)$
3. $\forall\beta(\beta < S(\alpha) \leftrightarrow \beta \leq \alpha)$

Beweis: Übung.

$$0 := \emptyset \qquad 1 := S(0) \qquad 2 := S(1) \quad \dots \quad n+1 := \{0, \dots, n\}$$

Definition: Eine Ordinalzahl α heißt:

- Nachfolgerordinalzahl (successor), wenn $\alpha = S(\beta)$ für ein β.
- Limesordinalzahl (limit), wenn $\alpha \neq 0$ und keine Nachfolgerordinalzahl.

Definition: α ist eine natürliche Zahl, wenn es keine Limesordinalzahl $\beta \leq \alpha$ gibt.
Bem.: Die bisherigen Axiome erlauben es nicht zu beweisen, dass die Menge aller natürlichen Zahlen existiert.

Ax. 7. Unendlichkeitsaxiom:

$$\exists x (0 \in x \wedge \forall y \in x (S(y) \in x))$$

Definition: $\omega := \{\alpha : \alpha \quad \text{ist eine natürliche Zahl}\}$

Bem.: Wegen $\alpha \in \beta \in \omega \Rightarrow \alpha \in \omega$ folgt, dass ω eine Ordinalzahl ist. Alle Ordinalzahlen kleiner als ω sind natürliche Zahlen, also entweder 0 oder eine Nachfolgerordinalzahl. Also ist ω die kleinste Limesordinalzahl.

1.4 Ordinalzahlarithmetik

Definition: $\alpha + \beta := type\langle \alpha \times \{0\} \cup \beta \times \{1\}, <_R \rangle$ mit

$$\langle \xi, i \rangle <_R \langle \eta, j \rangle \quad \text{gdw.} \quad \begin{cases} i = j & \text{und} \quad \xi < \eta \\ i = 0 & \text{und} \quad j = 1 \end{cases}$$

Lemma:

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha + 0 = \alpha$
3. $\alpha + 1 = S(\alpha)$
4. $\alpha + S(\beta) = S(\alpha + \beta)$
5. Wenn β eine Limesordinalzahl ist, dann gilt: $\alpha + \beta = sup\{\alpha + \xi, \xi < \beta\}$

Bem.: Die Summe ist *nicht* kommutativ:

$$\omega + 1 = S(\omega) > \omega$$
$$1 + \omega = sup\{1 + n : n < \omega\} = \omega$$

Definition: $\alpha \cdot \beta := type\langle \beta \times \alpha, <_R \rangle$ mit

$$\langle \xi, \eta \rangle <_R \langle \xi', \eta' \rangle \quad \text{gdw.} \quad \begin{cases} \xi < \xi' & \text{oder} \\ \xi = \xi' & \text{und} \quad \eta < \eta' \end{cases}$$

Lemma:

1. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
2. $\alpha \cdot 0 = 0$
3. $\alpha \cdot 1 = \alpha$
4. $\alpha \cdot S(\beta) = (\alpha \cdot \beta) + \alpha$
5. β Limes $\Rightarrow \alpha \cdot \beta = sup\{\alpha \cdot \xi : \xi < \beta\}$
6. $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$

Bem.:

- Kommutativität gilt nicht:

$$2 \cdot \omega = sup\{2 \cdot n : n < \omega\} = \omega$$

$$\omega \cdot 2 = \omega + \omega > \omega$$

- Distributivität gilt nicht in andere Richtung:

$$(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$$

$$(1 \cdot \omega) + (1 \cdot \omega) = \omega + \omega > \omega$$

Vorlesung am 07.05.03

1.5 Classes

We saw that $\{x : \Phi(x)\}$ may not be a set (e.g. for $\Phi(x) : x = x$). But there is nothing wrong with thinking about such collections. Informally, any collection of the form $\{x : \Phi(x)\}$ is called a *class*. A *proper class* is a class that does not form a set (because it is too "big").
Keep in mind that any subclass of a set is a set (by Comprehension).
Examples and Definition:

$$\mathbf{V} = \{x : x = x\}$$

$$\mathbf{ON} = \{x : x \text{ is an ordinal}\}$$

Both are *proper classes*. Formally, *proper classes* do not exist. An expression involving them must be thought as an abbreviation.
Ex.:

$$\mathbf{ON} = \mathbf{V} \quad \text{abbreviates} \quad \forall x(x \text{ is an ordinal} \leftrightarrow x = x)$$

In fact, there is no formal distinction between a class and a formula. We could think of $\mathbf{ON}$ as $\mathbf{ON}(x)$.

1.6 Recursion

Theorem 5: Transfinite induction on ON:
If $C \subseteq \mathbf{ON}$ and $C \neq \emptyset$ then C has a least element.

Proof:
Exactly like the proof of Theorem 2, 5. (same statements except that C was a set).
Take $\alpha \in C$: if α is not the least element of C, consider $\alpha \cap C$ (this exists by Comprehension). Let β be the least element of $\alpha \cap C$; this exists by Theorem 2, 5. β is the least element of C.□
There is a great difference between Theorem 2, 5. (where C is a set) and Theorem 5:

- Theorem 2, 5. is one *sentence* which is provable from ZFC.
- Theorem 5 is a theorem scheme which represents an infinite collection of theorems.

Theorem 5 allows "proofs by transfinite induction", i.e. one proves $\forall\alpha\varphi(\alpha)$ by showing for *each* α:

$$(\forall\beta < \alpha\varphi(\beta)) \rightarrow \varphi(\alpha).$$

Indeed, $\forall\alpha\varphi(\alpha)$ follows since $\exists\alpha\neg\varphi(\alpha)$ would imply $\neg\varphi(\alpha)$ holds for the least such α, a contradiction.
Sincerely, we can define a function of α recursively from information about the function below α (this is the same idea as those that occur in arithmetics, e.g. $0! = 1 \quad (n+1)! = n! \cdot (n+1)$).

Theorem 6: Transfinite Recursion on ON:
If $F : \mathbf{V} \longrightarrow \mathbf{V}$ then there is a unique

$$G : \mathbf{ON} \longrightarrow \mathbf{V} \quad \text{s.t.} \quad \forall\alpha[G(\alpha) = F(G \restriction \alpha)]$$

Formally:

- $F : \mathbf{V} \longrightarrow \mathbf{V}$ stands for any formula that says $\forall x \exists ! y \Phi_F(x, y)$,
- $G : \mathbf{ON} \longrightarrow \mathbf{V}$ stands for any formula that says $\forall x$ ordinal $\exists ! y \Phi_G(x, y)$.

Proof:

For *uniqueness*: if G_1 and G_2 both satisfy the condition, one proves $\forall \alpha (G_1(\alpha) = G_2(\alpha))$ by transfinite induction on α, i.e.: Assume this fails; take α the least such that $G_1(\alpha) \neq G_2(\alpha)$:

- $G_1(\alpha) = F(G_1 \restriction \alpha)$
- $G_2(\alpha) = F(G_2 \restriction \alpha)$
- Since $G_1 \restriction \alpha = G_2 \restriction \alpha$, this proves $G_1(\alpha) = G_2(\alpha)$, a contradiction.

For *existence*: Given ξ any ordinal, we say g is a ξ-approximation iff g is a function with domain ξ and $\forall \alpha < \xi [g(\alpha) = F(g \restriction \alpha)]$.

Now if g is a ξ-approximation and g' is a ξ'-approximation then $g \restriction (\xi \cap \xi') = g' \restriction (\xi \cap \xi')$ (same argument as for uniqueness of G). So there is a unique ξ-approximation, and if g' is a ξ'-approximation with $\xi < \xi'$ then $g' \restriction \xi = g$. It is enough to define G by $G(\alpha) = g(\alpha)$ for a ξ-approximation g for some (any) $\xi < \alpha$.□

Ex.: We could have defined $\alpha + \beta$ by recursion on β with the clauses:

- $\alpha + 0 = \alpha$
- $\alpha + S(\beta) = S(\alpha + \beta)$
- $\alpha + \beta = sup\{\alpha + \xi : \xi < \beta\}$, if β limit.

Formally, for each α define $F_\alpha : \mathbf{V} \longrightarrow \mathbf{V}$ by:

$$F_\alpha(x) = \begin{cases} 0 & \text{if } x \text{ is not a function with domain some ordinal} \\ \alpha & \text{if } x \text{ is a function with domain some ordinal } \beta = \emptyset \\ S(x(\vartheta)) & \text{if } x \text{ is a function with domain some ordinal } \beta = S(\vartheta) \\ \bigcup\{x(\xi) : \xi < \beta\} & \text{if } x \text{ is a function with domain some ordinal } \beta \text{ limit} \end{cases}$$

Theorem 6 yields a unique $G_\alpha : \mathbf{On} \longrightarrow \mathbf{V}$ s.t.

$$\forall \beta [G_\alpha(\beta) = F_\alpha(G_\alpha \restriction \beta)]$$

The uniqueness implies $\forall \alpha \forall \beta G_\alpha(\beta) = \alpha + \beta$.

Ex.: $\alpha \cdot \beta$ is defined by recursion on β by:

- $\alpha \cdot 0 = 0$
- $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$
- $\alpha \cdot \beta = sup\{\alpha \cdot \xi : \xi < \beta\}$ if β limit

These definitions of ordinal addition and multiplication coincide with the ones you saw yesterday.

Application: Ordinal Exponentiation

Definition: α^β is defined by recursion on β by

- $\alpha^0 := 1$
- $\alpha^{\beta+1} := \alpha^\beta \cdot \alpha$
- $\alpha^\beta := sup\{\alpha^\xi : \xi < \beta\}$ if β limit

Ex.:

$$2^\omega = sup\{2^\xi : \xi < \omega\} = sup\{2^n : n < \omega\} = \omega$$
$$2^{\omega+1} = 2^\omega \cdot 2 = \omega \cdot 2$$

1.7 Cardinals

Comparing the size of sets by use of 1-1 functions.

Definition:

1. $A \preceq B$ iff there exists a 1-1 function from A to B.
2. $A \approx B$ iff there exists a bijection between A and B.
3. $A \prec B$ if $A \preceq B \wedge B \not\preceq A$.

Ex.:

- $\mathbf{N} \approx \mathbf{Z}$
- $\mathbf{N} \approx \mathbf{N} \times \mathbf{N}$
- $\mathbf{N} \approx \mathbf{Q}$
- $\mathbf{N} \prec \mathbf{R}$

Theorem 7: Schröder-Bernstein

$$A \preceq B \wedge B \preceq A \rightarrow A \approx B$$

Proof: Assume $f : A \overset{1-1}{\longrightarrow} B, g : B \overset{1-1}{\longrightarrow} A$.
Set

$$A_0 = A, \quad B_0 = B; \qquad A_{n+1} = g''B_n, \quad B_{n+1} = f''A_n$$

It is clear that:

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$
$$B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$$
$$\text{Let} \quad A_\infty = \bigcap_{n\in\mathbf{N}} A_n \qquad B_\infty = \bigcap_{n\in\mathbf{N}} B_n$$

$$\text{We set:} \quad h(x) := \begin{cases} f(x) & \text{if} & x \in A_\infty \\ f(x) & \text{if} & x \in \bigcup_{n\in\mathbf{N}} A_{2n} \setminus A_{2n+1} \\ g^{-1}(x) & \text{otherwise} : & x \in \bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2} \end{cases}$$

h is well-defined since $x \in A_1 = g''B$.

1. $h \upharpoonright A_\infty$ is a bijection between A_∞ and B_∞.
2. $h \upharpoonright \bigcup_{n\in\mathbf{N}} A_{2n} \setminus A_{2n+1}$ is a bijection between

$$\bigcup_{n\in\mathbf{N}} A_{2n} \setminus A_{2n+1} \quad \text{and} \quad \bigcup_{n\in\mathbf{N}} B_{2n+1} \setminus B_{2n+2}$$

3. $h \restriction \bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2}$ is a bijection between

$$\bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2} \quad \text{and} \quad \bigcup_{n\in\mathbf{N}} B_{2n} \setminus B_{2n+1}$$

Remark:

$$g'' \bigcup_{n\in\mathbf{N}} B_{2n} \setminus B_{2n+1} = \bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2}$$

so that g is a bijection from

$$\bigcup_{n\in\mathbf{N}} B_{2n} \setminus B_{2n+1} \quad \text{to} \quad \bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2}$$

so that g^{-1} is a bijection from

$$\bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2} \quad \text{to} \quad \bigcup_{n\in\mathbf{N}} B_{2n} \setminus B_{2n+1.}$$

Finally:

$$\{A_\infty, \bigcup_{n\in\mathbf{N}} A_{2n} \setminus A_{2n+1}, \bigcup_{n\in\mathbf{N}} A_{2n+1} \setminus A_{2n+2.}\}$$

is a partition of A, and

$$\{B_\infty, \bigcup_{n\in\mathbf{N}} B_{2n} \setminus B_{2n+1}, \bigcup_{n\in\mathbf{N}} B_{2n+1} \setminus B_{2n+2.}\}.$$

is a partition of B. □

Vorlesung am 13.05.03

Definition: If A can be well-ordered, $|A|$ is the least α s.t. $\alpha \approx A$. $|A|$ is called the cardinal of α.

Ex.:

- A finite, is has 3 elements: $|A| = 3$
- $\mathbf{N}$: $|\mathbf{N}| = \omega$
- $|\mathbf{N} \times \mathbf{N}| = \omega$, since $\mathbf{N} \times \mathbf{N} \approx \mathbf{N} \approx \omega$.

Remark: If $\alpha \approx \beta$ then $|\alpha| = |\beta|$. Under AC, $|A|$ is defined for every set A.

Definition: α is a *cardinal*, iff $\alpha = |\alpha|$. Equivalently: α is a *carinal*, iff $\forall \beta < \alpha (\beta \not\approx \alpha)$.

Remark: If $|\alpha| \leq \beta \leq \alpha$ then $|\alpha| = |\beta|$.

Proof: $\beta \subseteq \alpha \rightarrow \beta \preceq \alpha$ and $|\alpha| \leq \beta \rightarrow |\alpha| \preceq \beta$.
By Schroeder-Bernstein-Theorem: $\beta \preceq \alpha \preceq |\alpha| \wedge |\alpha| \preceq \beta$.
So $\beta \approx |\alpha|$ so $|\beta| = |\alpha|$.□

Remark: If $n \in \omega$, then

1. $n \not\approx n + 1$
2. $\forall \alpha (\alpha \approx n \rightarrow \alpha = n)$.

Proof:

1. by induction on n
2. follows from previous remark.

Theorem: ω is a cardinal and each $n \in \omega$ is a cardinal.

Proof: Corollary of previous remark.

1.8 Cardinal Arithmetics

Definition:

1. $\kappa \oplus \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$
2. $\kappa \otimes \lambda = |\kappa \times \lambda|$.

Ex.:

- $\omega \oplus 1 = \omega \neq \omega + 1$
- $\omega \otimes 2 = \omega \neq \omega \cdot 2$.

Remark: $\oplus, \otimes$ are commutative.

Exercise:

1. $|\kappa + \lambda| = |\lambda + \kappa| = \kappa \oplus \lambda$
2. $|\kappa \cdot \lambda| = |\lambda \cdot \kappa| = \kappa \otimes \lambda$.

Remark: For $n, m \in \omega$

- $n \oplus m = n + m < \omega$

- $n \otimes m = n \cdot m < \omega$.

Proof: By induction on n.

Lemma. Every infinite cardinal is a limit ordinal.

Proof: If $\kappa = \alpha + 1$, then $1 + \alpha = \alpha$. $\kappa = |\kappa| = |\alpha + 1| = |1 + \alpha| = |\alpha|$, a contradiction.

Remark: The principle of transfinite induction can be applied to prove results about cardinal numbers, since every class of cardinals is a class of ordinals.

Theorem: If κ is infinite, then

$$\underbrace{\kappa \otimes \kappa}_{|\kappa \times \kappa|} = \kappa$$

Proof: By transfinite induction. So we assume this holds for smaller cardinals. For $\alpha < \kappa, |\alpha \times \alpha| = |\alpha| \otimes |\alpha| < \kappa$.
Define a well-ordering $\lhd$ on $\kappa \times \kappa$ by

$\langle \alpha, \beta \rangle \lhd \langle \gamma, \delta \rangle$ iff $[max(\alpha, beta) < max(\gamma, \delta) \vee (max(\alpha, \beta) = max(\gamma, \delta) \wedge \langle \alpha, \beta \rangle <_{lexic} \langle \gamma, \delta \rangle)]$

Remark that each $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ has no more than $|max(\alpha, \beta) + 1 \times max(\alpha, \beta) + 1| < \kappa$ predecessors.
So $type(\kappa \times \kappa, \lhd) \leq \kappa$, whence $|\kappa \times \kappa| \leq \kappa$. Since clearly $|\kappa \times \kappa| \geq \kappa$, one gets the result.

Definition: Given $n \in \omega$

A^n: the set of functions from n to A. Equivalently the set of sequences of exactly n elements of A.

$$A^{<\omega} = \bigcup_{n \in \mathbf{N}} A^n$$

Corollary: κ, λ be infinite cardinals:

1. $\kappa \oplus \lambda = \kappa \otimes \lambda = max(\kappa, \lambda)$.
2. $|\kappa^{<\omega}| = \kappa$.

Proof:

1. Assume $\kappa = max(\kappa, \lambda)$, then $\kappa \leq \kappa \oplus \lambda \leq \kappa \oplus \kappa \leq \kappa \otimes \kappa = \kappa$.
   $\kappa \leq \kappa \otimes \lambda \leq \kappa \otimes \kappa = \kappa$.
2. By previous Theorem there is a 1-1 mapping $f_2 : \kappa \times \kappa \longrightarrow \kappa$.
   Let f_1 be the identity $\kappa \longrightarrow \kappa$.
   Inductively, define $f_n : \kappa^n \longrightarrow \kappa$ that is 1-1 by

   $$\kappa^n \times \kappa \longrightarrow \kappa$$

   $$f_{n+1}(\langle \alpha_1, \ldots, \alpha_n \rangle) = \langle n, f_n(\langle \alpha_1, \ldots, \alpha_n \rangle) \rangle.$$

   It follows: $|\kappa^{<\omega}| \leq |\omega \times \kappa| \leq \omega \otimes \kappa = \kappa$.

Ax.: 8 : Power Set

$$\forall x \exists y \forall z (z \subseteq x \rightarrow z \in y)$$

Definition: (Power Set + Comprehension justify this definition)

$$\mathcal{P}(x) = \{z : z \subseteq x\}$$

Theorem (Cantor): For all sets x

$$x < \mathcal{P}(x)$$

Proof: $x \preceq \mathcal{P}(x)$ is obvious: $a \longrightarrow \{a\}$. To show $\mathcal{P}(x) \not\preceq x$ TAC we assume there exists $f : \mathcal{P}(x) \overset{1-1}{\longrightarrow} x$:

$$A = \{y \in ran(f) : y \notin f^{-1}(y)\}$$

Let $z = f(A), z \in A \leftrightarrow z \notin f^{-1}(z) = A$, a contradiction.□

Definition: α^+ is the least cardinal $> \alpha$.

- κ is a successor cardinal, iff $\kappa = \alpha^+$ for some α
- κ is a limit ordinal, iff $\kappa > \omega$ and κ is not a successor ordinal.

Definition: $\aleph_\alpha = \omega_\alpha$ is defined by transfinite recursion on α:

1. $\omega_0 = \omega$
2. $\omega_{\alpha+1} = (\omega_\alpha)^+$
3. For γ limit: $\omega_\gamma = sup\{\omega_\alpha : \alpha < \gamma\}$.

Lemma:

1. Each ω_α is a cardinal.
2. Every infinite cardinal is equal to ω_α for some α.
3. $\alpha < \beta \rightarrow \omega_\alpha < \omega_\beta$.
4. ω_α is a limit cardinal iff α is a limit ordinal. ω_α is a successor cardinal iff α is a successor ordinal.

Proof:

1. is obvious when α is a successor. When α is a limit: TAC assume ω_α is not a cardinal, then there exists some ordinal $\beta < \omega_\alpha$, s.t. $\beta \approx \omega_\alpha$. But $\beta < \omega_\alpha$ implies $\beta \leq \omega_\vartheta$ for some $\vartheta < \alpha$. Indeed $\beta < \omega_{\vartheta+1}$. But then $\beta < \omega_{\vartheta+1} < \omega_\alpha \preceq \beta$. A contradiction.
2. is immediate : TAC take the least infinite cardinal κ that is not equal to some ω_α. Consider $sup\{\alpha : \omega_\alpha < \kappa\} = \beta$, it satisfies $\omega_\beta = \kappa$.
3. is immediate by induction.
4. is immediate.

Vorlesung am 14.05.03

Definition:

- A is *finite* iff $|A| < \omega$.
- A is *countable* iff $|A| \leq \omega$.

Lemma: (AC): If there is a function f from X *onto* Y then $|Y| \leq |X|$.

Proof: Let $\langle X, <_R \rangle$ be a well-ordering, define $g : Y \longrightarrow X$ so that $g(y)$ is the $<_R$-least element of $f^{-1''}\{y\}$. Then $g : Y \longrightarrow X$ is 1-1 so $Y \preceq X$, so $|Y| \leq |X|$.□

Lemma: (AC) If $\kappa \geq \omega$ and $|X_\alpha| \leq \kappa$ for all $\alpha < \kappa$:

$$| \bigcup_{\alpha<\kappa} | \leq \kappa$$

Proof: For each α pick a 1-1 mapping $f_\alpha : X_\alpha \longrightarrow \kappa$ (these f_α are picked using a well-ordering of $\mathcal{P}(\bigcup_{\alpha<\kappa} X_\alpha \times \kappa)$). Define

$$f : \bigcup_{\alpha<\kappa} X_\alpha \overset{1-1}{\longrightarrow} \kappa \times \kappa \qquad f(x) = \langle \alpha, f_\alpha(x) \rangle,$$

where α is the least ordinal s.t. $x \in X_\alpha$. This shows that
$|\bigcup_{\alpha<\kappa} X_\alpha| \leq |\kappa \times \kappa| = \kappa \otimes \kappa = \kappa$.□

Application: Take $\kappa = \omega$. A countable union of countable sets is countable.

Levy showed that it is consistent with ZF (Set Theory without AC) that $\mathcal{P}(\omega)$ and also ω_1 are countable unions of countable sets.

1.9 Cardinal Exponentiation

Definition: ${}^B A = A^B = \{f : \text{ f is a function } \wedge dom(f) = B \wedge ran(f) \subseteq A\}$.
Notice that $A^B \subseteq \mathcal{P}(B \times A)$, so A^B exists by Power Set Axiom + Comprehension Scheme.

Remark: $2^B = \{0,1\}^B$ is the set of characteristic functions of subsets of B. In particular $2^B \approx \mathcal{P}(B)$.

Lemma: If $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$ then

$${}^A\kappa \approx^A 2 \approx \mathcal{P}(\lambda).$$

Proof: ${}^\lambda 2 \approx \mathcal{P}(\lambda)$ comes from identifying sets with their characteristic functions.

$${}^\lambda 2 \preceq^\lambda \kappa \preceq^\lambda \lambda \preceq \mathcal{P}(\lambda \times \lambda) \approx \mathcal{P}(\lambda) \approx^\lambda 2 \square.$$

Definition: (AC)

$$\kappa^\lambda = |{}^\lambda \kappa| = |\{f : f \text{ is a function } f : \lambda \longrightarrow \kappa\}|$$

Remark that $2^\lambda = |{}^\lambda 2| = |\mathcal{P}(\lambda)|$.

Ex.: If $|A| = n, |\mathcal{P}(A)| = 2^n$.
If $|A| = \omega = \omega_0 = \aleph_0, |\mathcal{P}(A)| = 2^\omega = 2^{\omega_0} = 2^{\aleph_0}$.

Lemma: (AC) If κ, λ, γ are cardinals:

1. $\kappa^{\lambda \oplus \gamma} = \kappa^\lambda \otimes \kappa\gamma$

2. $(\kappa^\lambda)^\gamma = \kappa^{\lambda \otimes \gamma}$.

Proof:

1. If $B \cap C = \emptyset$, then
$$^{(B \cup C)}A \approx ^B A \times ^C A$$
$$\varphi : f \longrightarrow \langle f \upharpoonright B, f \upharpoonright C \rangle \text{ is 1-1 and onto.}$$

2.
$$^C(^B A) \approx ^{(C \times B)} A$$
$$\psi : f \longrightarrow f'$$

   f is defined by $f(c) = g_c \in ^B A$.
   f' is defined by $f'(\langle c, b \rangle) = g_c(b)$.□

Where is the cardinality of $\mathbf{R}$? We can show:

$$|\mathbf{R}| = |\mathcal{P}(\omega)| = |^\omega 2| = 2^{\aleph_0}$$

Since Cantor could show that $2^{\omega_\alpha} \geq \omega_{\alpha+1}$ and had no way of producing cardinals between ω_α and 2^{ω_α} he conjectured that $2^{\omega_\alpha} = \omega_{\alpha+1}$.

Definition: (AC)

1. CH (the Continuum Hypothesis) is the statement $2^\omega = \omega_1$ ($2^{\aleph_0} = \aleph_1$).

2. GCH (the Generalized Continuum Hypothesis) is the statement
   $\forall \alpha (2^{\omega_\alpha} = \omega_{\alpha+1})$.

Later we will build a model of ZFC where GCH holds. $CH, \neg CH, GCH, \neg GCH$ are independent from ZFC, i.e. if ZFC is consistent $(CON(ZFC))$ then there is a model of $ZFC + CH, ZFC + \neg CH, ZFC + GCH, ZFC + \neg GCH$. In particular:

- $ZFC \nvdash CH$.
- $ZFC \nvdash GCH$.
- $ZFC \nvdash \neg CH$.
- $ZFC \nvdash \neg GCH$.

In 1941, Gödel proved: if $CON(ZFC)$ then there is a modelof $ZFC + GCH$.
In 1963, Cohen proved: if $CON(ZFC)$ then there is for each α a model of
$ZFC + 2^\omega = 2^{\aleph_0} = \aleph_{\alpha+1}$.

1.10 Cofinality

Definition: If $f : \alpha \longrightarrow \beta$, f maps α *cofinally* iff $ran(f)$ is unbounded in β (i.e. there is no $\vartheta \in \beta$ s.t. $\forall \eta \in \alpha f(\eta) < \vartheta$.

Ex.:
$f : \omega \longrightarrow \omega$ $\quad f(n) = \omega + n$.
$f : \omega \longrightarrow \omega^\omega$ $\quad f(n) = \omega^n + 3$.

Definition: The *cofinality* of β $(cf(b))$ is the least α, s.t. there is a cofinal map from α to β.

Remark: So $cf(\beta) \leq \beta$. And if β is a successor then $cf(\beta) = 1$.

Ex.:

- $cf(\omega) = \omega$
- $cf(\omega + \omega) = \omega$
- $cf(\omega^2) = \omega$
- $cf(\omega^{\omega^\omega}) = \omega$.
- $cf(\omega_1)$??

Vorlesung am 20.05.03

Lemma: There is a confinal map $f : cf(\beta) \longrightarrow \beta$ which is strictly *increasing*. $(\xi < \eta \rightarrow f(\xi) < f(\eta))$

Proof: Let $g : cd(\beta) \longrightarrow \beta$ be any cofinal map, and define f recursively by

$$f(\eta) = max(g(\eta), sup\{f(\xi) + 1 : \xi < \eta\}) \quad \Box$$

Lemma: If α is a limit ordinal and $f : \alpha \longrightarrow \beta$ is a strictly increasing cofinal map, then

$$cf(\alpha) = cf(\beta)$$

Proof:

1. $cf(\beta) \leq cf(\alpha)$ holds because

   $$cf(\alpha) \xrightarrow{g} \alpha \xrightarrow{f} \beta,$$ where g is a strictly increasing cofinal map.

   $f \circ g$ is a strictly increasing cofinal map.

2. $cf(\beta) \leq cf(\alpha)$ holds since:
   Let $g : cf(\beta) \longrightarrow \beta$ be a cofinal map, let $h(\xi)$ be the least η s.t. $f(\eta) > g(\xi)$ then $h : cf(\beta) \longrightarrow cf(\alpha)$ is a cofinal map.$\Box$

Corollary: $cf(cf(\beta)) = cf(\beta)$.

Definition: β is *regular* iff β is a limit ordinal and $cf(\beta) = \beta$. So by previous corollary, $cf(\beta)$ is regular for all limit ordinals β.

Lemma: If β is *regular*, then β is a cardinal.

Proof: If β is not a cardinal, take $\alpha = |\beta| < \beta$. Take any bijection $f : \alpha \longrightarrow \alpha$. It is a cofinal map which shows that $cf(\beta) < \beta$, a contradiction ($cf(\beta) = \beta$).

Lemma: ω is regular, obvious because ω is limit, and $cf(\omega) = \omega$.

Lemma: (AC): κ^+ is regular.

Proof: Is f mapped α cofinally into κ^+ where $\alpha < \kappa^+$ then

$$\kappa^+ = \bigcup\{f(\xi) : \xi < \alpha\}$$

But a union of $\leq \kappa$ set each of them of cardinality $\leq \kappa$ must have cardinality $\leq \kappa$, a contradiction.$\Box$
Limit cardinals often fail to be regular.

Ex.:

- $cf(\aleph_\omega) = \omega$
- $cf(\aleph_{\omega+\omega}) = \omega$
- $cf(\aleph_{\omega^\omega}) = \omega$.

Lemma: If α is a limit ordinal then

$$cf(\omega_\alpha) = cf(\aleph_\alpha) = cf(\alpha)$$

Proof: $\omega_\alpha = sup\{\omega_\xi : \xi < \alpha\}$ (by definition).
So $f : \alpha \longrightarrow \omega_\alpha$ defined by $f(\xi) = \omega_\xi$ is a strictly increasing cofinal map.$\Box$

As a consequence: if ω_α is regular, then $\omega_\alpha = \alpha$. But the condition $\omega_\alpha = \alpha$ is not sufficient.

For example: let $\gamma_0 = \omega, \gamma_{n+1} =^{\omega} \gamma_n =^{\aleph} \gamma_n$, and
$\alpha = sup\{\gamma_n : n \in \omega\} = sup\{\aleph_{\omega_{\omega_{\dots}}}\}$. Then α is the first ordinal to satisfy $\omega_\alpha = \alpha$, but $cf(\alpha) = \omega$.

Definition:

1. κ is *weakly inaccessible* iff κ is a regular limit cardinal.
2. AC: κ is *strongly inaccessible* iff $\kappa > \omega$, κ is regular and $\forall \lambda < \kappa(\underbrace{2^\lambda}_{|\mathcal{P}(\lambda)|} < \kappa)$.

Remark:

- If κ is strongly inaccessible $\longrightarrow$ κ is weakly inaccessible.
- If GCH holds then κ is strongly inaccessible iff κ is weakly inaccessible. $\forall \lambda(2^\lambda = \lambda^+)$.

1.11 Numbers

- $\mathbf{N} = \omega$
- $\mathbf{Z} = \omega \times \omega/_\sim$, where $\langle n, m\rangle \sim \langle n', m'\rangle$ iff $n - m = n' - m'$, i.e. $n + m' = n' + m$. So $\mathbf{Z}$ is represented by the equivalence class $\{\langle n, m\rangle : n - m = b\}$
- $\mathbf{Q} = (\mathbf{Z} \times (\mathbf{Z} \setminus \{0\}))/_\simeq$, where $\langle p, q\rangle \simeq \langle p', q'\rangle$ iff $\frac{p}{q} = \frac{p'}{q'}$, i.e. $pq' = p'q$. So a rational $\frac{p}{q}$ is represented by $\{\langle p', q'\rangle : \langle p', q'\rangle \simeq \langle p, q\rangle\}$.
- $\mathbf{R}$ = the set of left-sides of Dedekind cuts. Intuitively we represent a real π by
$$\pi : \quad]-\alpha, \pi[\quad \cap \mathbf{Q}$$
$$\mathbf{R} = \{X \in \mathcal{P}(\mathbf{Q}) : X \neq \emptyset \wedge X \neq \mathbf{Q} \wedge \forall x \in X \forall y \in \mathbf{Q}(y < x \rightarrow y \in X)\}$$

Fact:
$$\mathbf{N} \approx \mathbf{Z} \approx \mathbf{Q}$$
$$\mathbf{N} \approx \mathbf{N}^{<\omega} \approx \mathbf{Z}^{<\omega} \approx \mathbf{Q}^{<\omega}$$

They all have cardinality $\aleph_0 = \omega$.
$$\mathbf{R} \approx 2^{\mathbf{Q}} \approx \mathbf{N}^{\mathbf{N}} \approx 2^{\mathbf{N}}$$

They all have cardinality $2^{\aleph_0}$.

1.12 Well founded Sets

Or: How the Axiom of Foundation prohibits sets like $x = \{x\}$.

Definition: WF (well founded): the class of well founded sets is defined by transfinite recursion. Define $\mathbf{V}_\alpha$ for $\alpha \in \mathbf{ON}$ by:

1. $\mathbf{V}_0 = \emptyset$
2. $\mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V}_\alpha)$
3. $\mathbf{V}_\alpha = \bigcup_{\xi<\alpha} \mathbf{V}_\xi$, if α is a limit ordinal.

$$\mathbf{WF} = \bigcup\{\mathbf{V}_\alpha : \alpha \in \mathbf{ON}\}$$

Lemma: For each α:

1. $\mathbf{V}_\alpha$ is transitive ($x \in \mathbf{V}_\alpha \to x \subseteq \mathbf{V}_\alpha$).
2. $\forall \xi \leq \alpha(\mathbf{V}_\xi \leq \mathbf{V}_\alpha)$.

Proof: By transfinite induction on α we assume that this hypothesis holds for all $\beta < \alpha$ and we prove (1) and (2) holds at level α.

1. $\alpha := 0$ obvious
2. α is limit, (2) comes directly from the definition, (1) follows from the fact that the union of transitive sets is transitive.
3. $\alpha = \beta + 1$. Since $\mathbf{V}_\beta$ is transitive, $\mathcal{P}(\mathbf{V}_\beta) = \mathbf{V}_\alpha$ is transitive and $\mathbf{V}_\beta \subseteq \mathbf{V}_\alpha$. Hence (1) and (2).$\square$

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Remark that for any set $x \in \mathbf{WF}$, the least α for which $x \in \mathbf{V}_\alpha$ must be a successor ordinal.

Definition: If $x \in \mathbf{WF}$, $rank(x)$ is the least β s.t.

$$x \in \mathbf{V}_{\beta+1}$$

Ex.: $rank(\emptyset) = 0$.

So if $\beta = rank(x)$ then $x \subseteq \mathbf{V}_\beta$, $x \notin \mathbf{V}_\beta$ and $x \in \mathbf{V}_\alpha$ for any $\alpha < \beta$.

Lemma: For any α: $\mathbf{V}_\alpha = \{x \in \mathbf{WF} : rank(x) < \alpha\}$.

Proof: For $x \in \mathbf{WF}$

$$rank(x) < \alpha \quad \text{iff} \quad \exists \beta < \alpha (x \in \mathbf{V}_{\beta+1})$$

Lemma: If $y \in \mathbf{WF}$ then

1. $\forall x \in y (x \in \mathbf{WF} \wedge rank(x) \wedge rank(y))$
2. $rank(y) = sup\{rank(x) + 1 : x \in y\}$

Proof: For (1) let $\alpha = rank(y)$, then $y \in \mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V}_\alpha)$. If $x \in y$ then $x \in \mathbf{V}_\alpha$ so $rank(x) < \alpha$ by previous lemma.
For (2) let $\alpha = sup\{rank(x) + 1 : x \in y\}$; by (1) $\alpha \leq rank(y)$. And each $x \in y$ has $rank < \alpha$ so $y \subseteq \mathbf{V}_\alpha$. Thus $y \in \mathbf{V}_{\alpha+1}$. So $rank(y) \leq \alpha$.□

Lemma:

1. $\forall \alpha \in \mathbf{ON}(\alpha \in \mathbf{WF} \wedge rank(\alpha) = \alpha)$
2. $\forall \alpha \in \mathbf{ON}(\mathbf{V}_\alpha \cap \mathbf{ON} = \alpha)$.

Proof:

1. By transfinite induction on α. Assume that (1) holds for all $\beta < \alpha$. Then for $\beta < \alpha$, $\beta \in \mathbf{V}_{\alpha+1} \subseteq \mathbf{V}_\alpha$ so $\alpha \subseteq \mathbf{V}_\alpha$, so $\alpha \in \mathbf{V}_{\alpha+1}$.
2. is immediate from (1) and from the lemma that says $\mathbf{V}_\alpha = \{x \in \mathbf{WF} : rank(x) < \alpha\}$.

WF contains all ordinals, it is closed under all standard mathematical contructions: if $X, Y \in \mathbf{WF}$, then
$\bigcup X, \mathcal{P}(X), \{X\}, X \times Y, X \cup Y, X \cap Y, \{X, Y\}, \langle X, Y \rangle, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$ are all in **WF**.

Lemma: $\forall x (x \in \mathbf{WF} \leftrightarrow x \subseteq \mathbf{WF})$.

Proof: $'' \rightarrow''$: is just transitivity of $\in$ on **WF**.
$'' \leftarrow''$: If $X \subseteq \mathbf{WF}$, let $\alpha = sup\{rank(Y) + 1 : Y \in X\}$, then $X \subseteq \mathbf{V}_\alpha$, so $X \in \mathbf{V}_{\alpha+1}$.

What about the size of $\mathbf{V}_\alpha$?

Lemma: $\forall n \in \omega |\mathbf{V}_n| < \omega$.

Proof: by induction on n.

Lemma: $|\mathbf{V}_\omega| = \omega$.

Lemma (AC): $|\mathbf{V}_{\omega+\alpha}| = \Im_\alpha$. $\Im$ is defined by transfinite recursion:

1. $\Im_0 = \omega = \aleph_0$

2. $\Im_{\alpha+1} = 2^{\Im_\alpha}$

3. $\Im_\gamma = sup\{\Im_\gamma : \alpha < \gamma\}$ is γ limit.

$\langle A, R\rangle$ is a well-ordering, iff $\langle A, R\rangle$ is a total ordering s.t. any subset of A has a R-least element.

$$\begin{array}{ccc} \text{Well-orderings} & \longleftrightarrow & \text{ordinals} \\ \text{Well-founded relations} & \longleftrightarrow & \text{Well-founded Sets (} \mathbf{WF}) \end{array}$$

1.13 Well-Founded Relations

This is a generalization of the notation of well-ordering.

Definition: A relation R is well-founded on a set A iff

$$\forall X \subseteq A[X \neq \emptyset \rightarrow \exists y \in X(\neg\exists z \in X(zRy))]$$

This y is called an R-minimum in X. So R is well-founded on A, if every non-empty subset of A has an R-minimal element. In particular if R is a total ordering on A then R well-founded $\leftrightarrow$ R well-orders A.

Ex.: Be A any set, $A^{<\omega}$ = set of all finite sequences of elemetnts of A.
R is defined by: for $x, y \in A^{<\omega}$, xRy iff x is a strict prefix of y (i.e. x is a strict initial set of y).

Lemma: If $A \in \mathbf{WF}$, $\in$ is well-founded on A.

Proof: Let X be a non-empty subset of A. Let $\alpha = min\{rank(y) : y \in X\}$ and take $y \in X$ with $rank(y) = \alpha$. Then y is $\in$-minimal in X (because otherwise if $\exists x \in X \wedge x \in y$ then $rank(x) < rank(y) = \alpha$, a contradiction).

Lemma: If A is transitive and $\in$ is well-founded on A then $A \in \mathbf{WF}$.

Proof: It is enough to show that $A \subseteq \mathbf{WF}$. If $A \not\subseteq \mathbf{WF}$, let $X = A - \mathbf{WF} \neq \emptyset$. Take y a $\in$-minimal element in X. If $z \in y$ then $z \notin X$, but $z \in A$ (since A is transitive), so $z \in \mathbf{WF}$.
Thus $y \subseteq \mathbf{WF}$, so $y \in \mathbf{WF}$, a contradiction.□

Definition: By recursion on n define $\bigcup^n A$ by:

1.
$$\bigcup\nolimits^0 A = A$$
$$\bigcup\nolimits^{n+1} A = \bigcup(\bigcup\nolimits^n A)$$

2.
$$\underbrace{trcl(A)}_{\text{transitive closure of A}} = \bigcup\{\bigcup\nolimits^n A : n \in \omega\}$$

$$trcl(A) = \underbrace{A}_{\text{elements of A}} \cup \underbrace{\bigcup\nolimits^1 A}_{\text{elements of elements of A}} \cup \bigcup\nolimits^2 A \cup \bigcup\nolimits^3 A \cup \ldots$$

Lemma:

1. $A \subseteq trcl(A)$

2. $trcl(A)$ is transitive

3. If $A \subseteq T$ and T is transitive then $trcl(A) \subseteq T$.
4. If A is transitive, then $trcl(A) = A$
5. If $x \in A$, then $trcl(x) \subseteq trcl(A)$
6. $trcl(A) = A \cup \bigcup\{trcl(x) : x \in A\}$.

Proof:

1. Obvious.
2. Use $y \in \bigcup^n A \to y \in \bigcup^{n+1} A$.
3. By induction on n we show that $\bigcup^n A \subseteq T$.
4. Follows from (1) and (3) taking $A = T$.
5. $x \in A \to x \in trcl(A) \to x \subseteq trcl(A)$, so apply (3) to x.
6. Let $T = A \cup \bigcup\{trcl(x) : x \in A\}$. T is transitive, so $trcl(A) \subseteq T$ by (3). But $T \subseteq trcl(A)$ by (1) and (5). □

Theorem: For all sets A the following are equivalent:

1. $A \in \mathbf{WF}$
2. $trcl(A) \in \mathbf{WF}$
3. $\in$ is well-founded on $trcl(A)$.

Proof: ((1) → (2)): If $A \in \mathbf{WF}$, then by induction on n $\bigcup^n A \in \mathbf{WF}$, since $\mathbf{WF}$ is closed under $\bigcup$. Then each $\bigcup^n A \subseteq \mathbf{WF}$, so $trcl(A) \subseteq \mathbf{WF}$, so $trcl(A) \in \mathbf{WF}$.
((2) → (3)): already proved.
((3) → (1)): Since $\in$ is well-founded on $trcl(A)$, $trcl(A)$ is transitive, $trcl(A) \in \mathbf{WF}$. So $A \subseteq trcl(A) \subseteq \mathbf{WF}$, so $A \subseteq \mathbf{WF}$, so $A \in \mathbf{WF}$.

1.14 The axiom of Foundation

Ax. 2:

$$\forall x(\exists y(y \in x) \to \exists(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$$

Equivalently: if $x \neq \emptyset$, then $\exists y \in x(x \cap y = \emptyset)$.
Or: Every non-empty set has an $\in$-minimal element.

Theorem: The following are equivalent (FAE):

1. The Axiom of Foundation
2. $\forall A(\in$ is well-founded on $A)$
3. $\mathbf{V} = \mathbf{WF}$

Proof: ((1) → (2)): is immediate from the definition of well-foundedness.
((2) → (3)): (2) implies for any set A, $\in$ is well-founded on $trcl(A)$, so $A \in \mathbf{WF}$.
((3) → (2)): By lemma that says if $A \in \mathbf{WF}$ then $\in$ is well-founded on A.□