Teillösung 9. Übung

12. Juli 2002

Aufgabe 32

b)

$$X \sim \overline{\mathrm{Bin}} = \underbrace{\mathrm{Geo}(p) * \ldots * \mathrm{Geo}(p)}_{n}$$

$$Y_i \sim \mathrm{Geo}(p) \quad i = 1, \ldots, n, \quad \text{s.u.}, \quad X = \sum_{i=1}^{n} Y_i$$

$$\mathrm{E}(X) = \mathrm{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathrm{E}\left(Y_i\right) = n\,\mathrm{E}\left(Y_1\right) = \frac{n(1-p)}{p} =: n\frac{q}{p}$$

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) \underset{\text{s.u.}}{=} \sum_{i=1}^{n} \mathrm{Var}\left(Y_i\right) = n\,\mathrm{Var}\left(Y_1\right)$$

Bezeichnung: $Y = Y_1$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \left(\mathrm{E}(Y)\right)^2$$

$$\mathrm{E}(Y^2) = \mathrm{E}(Y(Y-1)+Y) = \mathrm{E}(Y(Y-1)) + \mathrm{E}(Y) = \sum_{k=0}^{\infty} k(k-1)q^k p + \frac{q}{p}$$

$$= pq^2 \sum_{k=2}^{\infty} k(k-1)q^{k-2} + \frac{q}{p} = pq^2 \sum_{k=2}^{\infty} \frac{d^2}{dq^2} q^k + \frac{q}{p} = pq^2 \frac{d^2}{dq^2} \sum_{k=2}^{\infty} q^k + \frac{q}{p}$$

$$= pq^2 \frac{d^2}{dq^2} \frac{q^2}{1-q} + \frac{q}{p} = pq^2 \frac{d}{dq} \frac{2q-q^2}{(1-q)^2} + \frac{q}{p} = 2\frac{q^2}{p^2} + \frac{q}{p} = 2\frac{(1-p)^2}{p^2} + \frac{1-p}{p}$$

$$\Rightarrow \quad \mathrm{Var}(Y) = 2\frac{(1-p)^2}{p^2} + \frac{1-p}{p} - \frac{(1-p)^2}{p^2} = \frac{1-p}{p^2}$$

$$\Rightarrow \quad \mathrm{Var}(X) = n\frac{1-p}{p^2}$$

c) $X \sim \Gamma(\alpha, \lambda)$
Berechne die Laplace-Transformierte:

$$\begin{aligned}
L(s) &= \int_0^\infty e^{-sx} f(x)\, dx \\
&= \int\limits_0^\infty e^{-sx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int\limits_0^\infty x^{\alpha-1} e^{-(\lambda+s)x}\, dx \\
&\quad \text{Substituiere: } (\lambda+s)x = y \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{y}{\lambda+s}\right)^{\alpha-1} e^{-y} \frac{1}{\lambda+s}\, dy = \frac{\lambda^\alpha}{(\lambda+s)^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty y^{\alpha-1} e^{-y}\, dy}_{\Gamma(\alpha)} \\
&= \left(\frac{\lambda}{\lambda+s}\right)^\alpha
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X) &= -L^{'}(0) = -\left(\alpha\left(\frac{\lambda}{\lambda+s}\right)^{\alpha-1}\left(-\frac{\lambda}{(\lambda+s)^2}\right)\right)\Bigg|_{s=0} = \frac{\alpha}{\lambda} \\
\mathrm{E}(X^2) &= L^{''}(0) = \left(-\alpha\frac{\lambda^\alpha}{(\lambda+s)^{\alpha+1}}\right)^{'}\Bigg|_{s=0} \\
&= \left(-\alpha\lambda^\alpha(-\alpha-1)\frac{1}{(\lambda+s)^{\alpha+2}}\right)\Bigg|_{s=0} = \frac{\alpha(\alpha+1)}{\lambda^2} \\
&\Rightarrow \quad \mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{\alpha}{\lambda^2}
\end{aligned}$$

d)

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-t}^t \frac{x}{\pi(1+x^2)}\, dx \\
&= \left[\ln\frac{1}{2}(1+x^2)\right]_{-t}^t \\
&\Rightarrow \quad \lim_{t\to\infty} \int_{-t}^t x f_x(x)\, dx = 0 \\
\text{Aber} &: \quad \mathrm{E}(|X|) = 2\int_0^\infty \frac{x}{\pi(1+x^2)}\, dx = \infty \\
&\Rightarrow \quad \text{Der Erwartungswert existiert nicht.} \\
&\Rightarrow \quad \text{Die Varianz existiert ebenfalls nicht.}
\end{aligned}$$

Aufgabe 33

a) X, Y absolut-stetig: siehe Skript.
X, Y diskret, mit Träger T_X, T_Y:

$$\begin{aligned}
\mathrm{E}(aX+bY) &= \sum_{x\in T_X}\sum_{y\in T_Y}(ax+by)P(X=x,Y=y)\\
&= a\sum_{x\in T_X, y\in T_Y} x\,P(X=x,Y=y) + b\sum_{x\in T_X, y\in T_Y} y\,P(X=x,Y=y)\\
&= a\sum_{x\in T_X} x\,P(X=x) + b\sum_{y\in T_Y} y\,P(Y=y)\\
&= a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)
\end{aligned}$$

b) $X \leq Y \quad \Rightarrow \quad Y - X \geq 0$

$$\Rightarrow \quad \mathrm{E}(Y-X) = \begin{cases} \displaystyle\sum_{X\in T_X, Y\in T_Y} (y-x)P(X=x,Y=y) \ \geq \ 0 \\ \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-x) f_{(X,Y)}(x,y) \ \geq \ 0 \end{cases}$$

$$\underset{\text{mit a)}}{\Rightarrow} \quad \mathrm{E}(X) - \mathrm{E}(Y) \geq 0 \quad \Rightarrow \quad \mathrm{E}(Y) \geq E(X).$$